MATH 10550
EXAM III SOLUTIONS

1. Solving the equation $x^2 - 2 + \cos(\frac{\pi x}{2}) = 0$ using Newton's method with initial approximation $x_1 = 1$, what is x_2?

Solution. Let $f(x) = x^2 - 2 + \cos(\frac{\pi x}{2})$. Then

$$\begin{aligned}
&x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \\
&f(x_1) = f(1) = -1 \\
&f'(x) = 2x - \frac{\pi}{2}\sin(\frac{\pi x}{2}) \\
&f'(x_1) = f'(1) = 2 - \frac{\pi}{2} = \frac{4-\pi}{2} \\
&\text{Therefore } x_2 = 1 + \frac{2}{4-\pi} = \frac{6-\pi}{4-\pi}
\end{aligned}$$

2. The area of an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

of semi-axis a and b is known to be πab. Use this (or some other geometric fact) to evaluate the integral

$$\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}dx$$

Solution. Solve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for y.

$$\begin{aligned}
\frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1 \\
x^2 + \frac{a^2}{b^2}y^2 &= a^2 \quad \text{(multiply by } a^2\text{)} \\
\frac{a^2}{b^2}y^2 &= a^2 - x^2 \quad \text{(subtract } x^2\text{)} \\
\frac{a}{b}y &= \sqrt{a^2 - x^2} \quad \text{(take the positive square root)} \\
y &= \frac{b}{a}\sqrt{a^2 - x^2}
\end{aligned}$$

It is now clear that $\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}dx$ is the area of the top half of the ellipse with equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore $\int_{-a}^{a} \frac{b}{a}\sqrt{a^2 - x^2}dx = \frac{1}{2}\pi ab$.

3. Calculate the indefinite integral

$$\int \frac{3x + 3\sqrt{x}}{\sqrt{x}} dx =$$

Solution.

$$\begin{aligned} \int \frac{3x + 3\sqrt{x}}{\sqrt{x}} dx &= \int 3\sqrt{x} + 3dx \\ &= 2x^{\frac{3}{2}} + 3x + C \end{aligned}$$

4. Calculate the definite integral

$$\int_0^\pi |\cos x| dx =$$

Solution.

$$\begin{aligned} \int_0^\pi |\cos x| dx &= \int_0^{\frac{\pi}{2}} \cos x dx + \int_{\frac{\pi}{2}}^\pi - \cos x dx \\ &= [\sin x]_0^{\frac{\pi}{2}} + [-\sin x]_{\frac{\pi}{2}}^\pi \\ &= \left(\sin \frac{\pi}{2} - \sin 0\right) + \left(-\sin \pi + \sin \frac{\pi}{2}\right) \\ &= (1 - 0) + (0 + 1) \\ &= 2 \end{aligned}$$

5. Calculate

$$\int 6 \tan^5 x \sec^2 x dx =$$

Solution. Use substitution with $u = \tan x$. Then $\frac{du}{dx} = sec^2 x$.

$$\begin{aligned} \int 6 \tan^5 x \sec^2 x dx &= \int 6u^5 du \\ &= u^6 + C \\ &= \tan^6 x + C \end{aligned}$$

6. Which of the following estimates hold for the integral

$$I = \int_0^1 (1 + \cos^2 x) dx?$$

Solution. Note $1 \leq 1 + \cos^2 x \leq 2$ for any x. Consequently $\int_0^1 1 dx \leq I \leq \int_0^1 2 dx$. Therefore $1 \leq I \leq 2$.

7. Find the volume of the solid obtained by rotating the region bounded by $y = x^6$, $y = 1$, and $x = 0$, about the y - axis.
Solution. Use disks with radius $r = y^{\frac{1}{6}}$. The area of each disk is then $\pi y^{\frac{1}{3}}$.

$$V = \int_0^1 \pi y^{\frac{1}{3}} dy = \left[\frac{3\pi}{4} y^{\frac{4}{3}}\right]_0^1 = \frac{3\pi}{4}$$

8. Consider the function

$$g(x) = -\int_{\sin x}^{0} \sqrt{t^3+1}dt.$$

Then $g'(x) =$
Solution. Note first that $-\int_{\sin x}^{0} \sqrt{t^3+1}dt = \int_0^{\sin x} \sqrt{t^3+1}dt$. By the Fundamental Theorem of Calculus (and the chain rule)
$g'(x) = \cos x\sqrt{\sin^3 x + 1}$.

9. Calculate the integral

$$\int_{-2}^{2} \frac{x^3}{1+\cos^2 x} dx.$$

Solution. Finding an antiderivative for the integrand would be challenging. It is easier to check that $\frac{x^3}{1+\cos^2 x}$ is an odd function. Therefore $\int_{-2}^{2} \frac{x^3}{1+\cos^2 x} dx = 0$.

10. Which of the following is a Riemann sum corresponding to the integral

$$\int_1^2 \sin x \; dx?$$

Solution. Only one of the options is a Riemann sum.
$\frac{1}{n}\sum_{i=1}^{n} \sin(1+\frac{i}{n})$ is the Riemann sum obtained by evaluating $\sin x$ at the right endpoints of the sub-intervals.

11. Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$ and the vertical lines $x = 0$, $x = \frac{\pi}{2}$.
Solution. This problem is Example 5 on page 350 of the textbook.

12. Find the coordinates of the point on the line $x + y + 1 = 0$ that is closest to the origin. Hint: the computations are a bit easier if you minimize the <u>*square*</u> of the distance to the origin.

Solution. Solve $x + y + 1 = 0$ for y to find $y = -x - 1$. Minimize the function $D = x^2 + y^2$ given that $x + y + 1 = 0$. First write D as a function of one variable by substituting $y = -x - 1$. So

$$D = x^2 + (-x - 1)^2 = 2x^2 + 2x + 1.$$

Find the critical points. $D' = 4x + 2 = 0$ gives $x = -\frac{1}{2}$. Use the first (or second) derivative test to check that this is a local minimum. The point on the line closest to the origin is $(-\frac{1}{2}, -\frac{1}{2})$.

13. A cylindrical can without a top is made to contain π cubic centimeters of liquid. Find the dimensions (height and radius of the cylinder) that will minimize the cost of the metal to make the can.
Solution. The surface area of the can is given by

$$SA = \pi r^2 + 2\pi r h.$$

The volume of the can is given by $V = \pi r^2 h$. Use $V = \pi$ to find that $h = \frac{1}{r^2}$ (we know $r \neq 0$). Substitute to find

$$SA = \pi r^2 + \frac{2\pi}{r}.$$

Find the critical points.

$$\begin{aligned} 0 &= \frac{dSA}{dr} = 2\pi r - \frac{2\pi}{r^2} \\ 0 &= r^3 - 1 \\ r &= 1 \end{aligned}$$

Use the first (or second) derivative test to check that this is local minimum. The optimized can has $r = 1$ and $h = 1$.